ON DIFFERENTIAL GEOMETRY OF HYPERSURFACES IN THE LARGE

BY

CHUAN-CHIH HSIUNG

1. **Introduction.** Let V^n (V^{*n}) be an orientable hypersurface of class C^3 imbedded in a Euclidean space E^{n+1} of $n+1\geqq 3$ dimensions with a closed boundary V^{n-1} (V^{*n-1}) of dimension $n-1$. Suppose that there is a one-to-one correspondence between the points of the two hypersurfaces V^n, V^{*n} such that at corresponding points the two hypersurfaces V^n, V^{*n} have the same normal vectors. Let $\kappa_1, \cdots, \kappa_n$ be the n principal curvatures at a point P of the hypersurface V^n, then the αth mean curvature M_α of the hypersurface V^n at the point P is defined by

$$\binom{n}{\alpha} M_\alpha = \sum \kappa_1 \cdots \kappa_\alpha \qquad (\alpha = 1, \cdots, n), \tag{1.1}$$

where the expression on the right side is the αth elementary symmetric function of $\kappa_1, \cdots, \kappa_n$. In particular, M_n is the Gaussian curvature of the hypersurface V^n at the point P. Let P^* be the point of the hypersurface V^{*n} corresponding to the point P of the hypersurface V^n under the given correspondence, p^* the oriented distance from a fixed point O in the space E^{n+1} to the tangent hyperplane of the hypersurface V^{*n} at the point P^*, and dA the area element of the hypersurface V^n at the point P. The purpose of this paper is first to derive some expressions for the integrals $\int_{V^n} M_\alpha p^* dA$ ($\alpha=1, \cdots, n$), and then to prove the following

THEOREM. *Let V^n (V^{*n}) be an orientable hypersurface of class C^3 imbedded in a Euclidean space E^{n+1} of $n+1\geqq 3$ dimensions with a positive Gaussian curvature and a closed boundary V^{n-1} (V^{*n-1}) of dimension $n-1$. Suppose that there is a one-to-one correspondence between the points of the two hypersurfaces V^n, V^{*n}, such that at corresponding points the two hypersurfaces V^n, V^{*n} have the same normal vectors and equal sums of the principal radii of curvature, and such that the two boundaries V^{n-1}, V^{*n-1} are congruent. Then the two hypersurfaces V^n, V^{*n} are congruent or symmetric.*

This theorem has been obtained by T. Kubota (see [6] or [1, pp. 29–30]) for closed hypersurfaces V^n, V^{*n}, and by the author [5] for $n=2$ in a slightly different form.

2. **Preliminaries.** In a Euclidean space E^{n+1} of dimension $n+1\geqq 3$, let us

Received by the editors May 8, 1955.

consider a fixed orthogonal frame $O\mathfrak{Y}_1 \cdots \mathfrak{Y}_{n+1}$ with a point O as the origin. With respect to this orthogonal frame we define the vector product of n vectors $A_1, \cdots, A_n$ in E^{n+1} to be the vector A_{n+1}, denoted by $A_1 \times \cdots \times A_n$, satisfying the following conditions:

(a) the vector A_{n+1} is normal to the n-dimensional space determined by the vectors $A_1, \cdots, A_n$,

(b) the magnitude of the vector A_{n+1} is equal to the volume of the parallelepiped whose edges are the vectors $A_1, \cdots, A_n$,

(c) the two frames $OA_1 \cdots A_n A_{n+1}$ and $O\mathfrak{Y}_1 \cdots \mathfrak{Y}_{n+1}$ have the same orientation.

Let σ be a permutation on the n numbers $1, \cdots, n$, then

$$A_{\sigma(1)} \times \cdots \times A_{\sigma(n)} = (\operatorname{sgn} \sigma) A_1 \times \cdots \times A_n, \tag{2.1}$$

where $\operatorname{sgn} \sigma$ is $+1$ or -1 according as the permutation σ is even or odd. Let $i_1, \cdots, i_{n+1}$ be the unit vectors from the origin O in the directions of the vectors $\mathfrak{Y}_1, \cdots, \mathfrak{Y}_{n+1}$ and let A^j_α $(j=1, \cdots, n+1)$ be the components of the vector A_α $(\alpha=1, \cdots, n)$(1) with respect to the frame $O\mathfrak{Y}_1 \cdots \mathfrak{Y}_{n+1}$, then the scalar product of any two vectors A_α and A_β and the vector product of n vectors $A_1, \cdots, A_n$ are, respectively,

$$A_\alpha \cdot A_\beta = \sum_{i=1}^{n+1} A^i_\alpha A^i_\beta, \tag{2.2}$$

$$A_1 \times \cdots \times A_n = (-1)^n \begin{vmatrix} i_1 & i_2 & \cdots & i_{n+1} \\ A^1_1 & A^2_1 & \cdots & A^{n+1}_1 \\ \cdot & \cdot & \cdot & \cdot \\ A^1_n & A^2_n & \cdots & A^{n+1}_n \end{vmatrix}. \tag{2.3}$$

If A^j_α are differentiable functions of n variables $x^1, \cdots, x^n$, then by equation (2.3) and the differentiation of determinants

$$\frac{\partial}{\partial x^\alpha}(A_1 \times \cdots \times A_n) = \sum_{\beta=1}^{n}\left(A_1 \times \cdots \times A_{\beta-1} \times \frac{\partial A_\beta}{\partial x^\alpha} \times A_{\beta+1} \times \cdots \times A_n\right). \tag{2.4}$$

Now we consider a hypersurface V^n of class C^3 imbedded in the space E^{n+1} with a closed boundary V^{n-1} of dimension $n-1$. Let $(y^1, \cdots, y^{n+1})$ be the coordinates of a point P in the space E^{n+1} with respect to the orthogonal frame $O\mathfrak{Y}_1 \cdots \mathfrak{Y}_{n+1}$. Then the hypersurface V^n can be given by the parametric equations(2)

(1) Throughout this paper all Latin indices take the values 1 to $n+1$ and Greek indices the values 1 to n unless stated otherwise. We shall also follow the convention that repeated indices imply summation.

(2) For the remainder of this section see, for instance, [2, Chap. IV].

(2.5) $$y^i = f^i(x^1, \cdots, x^n) \qquad (i = 1, \cdots, n+1),$$

or the vector equation

(2.6) $$Y = F(x^1, \cdots, x^n),$$

where y^i and f^i are respectively the components of the two vectors Y and F, the parameters $x^1, \cdots, x^n$ take values in a simply connected domain D of the n-dimensional real number space, $f^i(x^1, \cdots, x^n)$ are of the third class and the Jacobian matrix $\|\partial y^i/\partial x^\alpha\|$ is of rank n at all points of D. If we denote the vector $\partial Y/\partial x^\alpha$ by Y_α for $\alpha = 1, \cdots, n$, then the first fundamental form of the hypersurface V^n at a point P is

(2.7) $$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta,$$

where

(2.8) $$g_{\alpha\beta} = Y_\alpha \cdot Y_\beta,$$

and the matrix $\|g_{\alpha\beta}\|$ is positive definite so that the determinant $g = |g_{\alpha\beta}| > 0$.

Let N be the unit normal vector of the hypersurface V^n at a point P, and N_α the vector $\partial N/\partial x^\alpha$, then

(2.9) $$N_\alpha = -b_{\alpha\beta}g^{\beta\gamma}Y_\gamma,$$

where

(2.10) $$b_{\alpha\beta} = b_{\beta\alpha} = -Y_\alpha \cdot N_\beta$$

are the coefficients of the second fundamental form of the hypersurface V^n at the point P, and $g^{\beta\gamma}$ denotes the cofactor of $g_{\beta\gamma}$ in g divided by g so that

(2.11) $$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma},$$

δ^α_γ being the Kronecker deltas. The n principal curvatures $\kappa_1, \cdots, \kappa_n$ of the hypersurface V^n at the point P are the roots of the determinant equation

(2.12) $$| b_{\alpha\beta} - \kappa g_{\alpha\beta} | = 0.$$

From equations (1.1) and (2.12) follow immediately

(2.13) $$M_n = b/g, \qquad nM_1 = b_{\alpha\beta}g^{\alpha\beta}, \qquad nM_{n-1} = g_{\alpha\beta}B^{\alpha\beta}/g,$$

where $b = |b_{\alpha\beta}| \neq 0$ and $B^{\alpha\beta}$ is the cofactor of $b_{\alpha\beta}$ in b.

The third fundamental form of the hypersurface V^n at the point P is

(2.14) $$dN \cdot dN = f_{\alpha\beta}dx^\alpha dx^\beta,$$

where we have placed

(2.15) $$f_{\alpha\beta} = N_\alpha \cdot N_\beta.$$

From equations (2.8), (2.9), and (2.11), it follows immediately that

$$f_{\alpha\beta} = b_{\alpha\rho}b_{\beta\sigma}g^{\rho\sigma}, \tag{2.16}$$

and therefore that

$$g^{\alpha\beta} = f_{\rho\sigma}b^{\alpha\rho}b^{\beta\sigma}, \tag{2.17}$$

where $b^{\alpha\rho}=B^{\alpha\rho}/b$. It is easily seen that the principal radii of curvature r_α ($\alpha=1, \cdots, n$) of the hypersurface V^n at the point P are the roots of the determinant equation

$$\left| b_{\alpha\beta} - rf_{\alpha\beta} \right| = 0, \tag{2.18}$$

from which we obtain

$$r_1 \cdots r_n = b/f, \qquad \sum_{\alpha=1}^{n} r_\alpha = b_{\alpha\beta}f^{\alpha\beta}, \qquad \sum r_1 \cdots r_{n-1} = f_{\alpha\beta}B^{\alpha\beta}/f, \tag{2.19}$$

where $f^{\alpha\beta}$ denotes the cofactor of $f_{\alpha\beta}$ in $f=\left|f_{\alpha\beta}\right|$ divided by f. From equations (2.13) and (2.19) it follows immediately that

$$f = M_n^2 g > 0. \tag{2.20}$$

The area element of the hypersurface V^n at the point P is given by

$$dA = g^{1/2}dx^1 \cdots dx^n. \tag{2.21}$$

Now we choose the direction of the unit normal vector N in such a way that the two frames $PY_1 \cdots Y_nN$ and $O\mathfrak{Y}_1 \cdots \mathfrak{Y}_{n+1}$ have the same orientation. Then from equations (2.3) and (2.21) it follows that

$$g^{1/2}N = Y_1 \times \cdots \times Y_n, \tag{2.22}$$

$$\left| Y_1, \cdots, Y_n, N \right| = g^{1/2}. \tag{2.23}$$

Let $u^1, \cdots, u^{n-1}$ be the local coordinates of a point P on the boundary V^{n-1}, then the first fundamental form of the boundary V^{n-1} at the point P is

$$ds^2 = a_{\lambda\mu}du^\lambda du^\mu \qquad (\lambda, \mu = 1, \cdots, n-1), \tag{2.24}$$

where

$$a_{\lambda\mu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial u^\lambda}\frac{\partial x^\beta}{\partial u^\mu}, \tag{2.25}$$

and the matrix $\|a_{\lambda\mu}\|$ is positive definite so that the determinant $a=\left|a_{\lambda\mu}\right|>0$. The coefficients of the second fundamental form of the boundary V^{n-1} corresponding to the unit normal vector N of the hypersurface V^n at the point P are

$$\Omega_{\lambda\mu} = \sum_{i=1}^{n+1} N^i \left(\frac{\partial^2 y^i}{\partial u^\lambda \partial u^\mu} - \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}_a \cdot \frac{\partial y^i}{\partial u^\nu} \right) \qquad (\lambda, \mu, \nu = 1, \cdots, n-1), \tag{2.26}$$

where

$$\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}_a$$

is a Christoffel symbol of the second kind formed with respect to the a's and u's. Similarly, for the hypersurface V^n we have

$$b_{\alpha\beta} N = \frac{\partial^2 Y}{\partial x^\alpha \partial x^\beta} - \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}_g Y_\gamma, \tag{2.27}$$

where

$$\left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}_g$$

is a Christoffel symbol of the second kind formed with respect to the g's and x's. From equations (2.26) and (2.27) it is easily seen that

$$\Omega_{\lambda\mu} = b_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^\lambda} \frac{\partial x^\beta}{\partial u^\mu} \cdot \tag{2.28}$$

3. **Some integral formulas.** Suppose that there is a one-to-one correspondence between the points of two hypersurfaces V^n, V^{*n} of class C^3 imbedded in a space E^{n+1} with positive Gaussian curvatures and closed boundaries V^{n-1}, V^{*n-1} of dimension $n-1$ respectively such that the two hypersurfaces V^n, V^{*n} have the same normal vectors at corresponding points. Without loss of generality we may assume that the corresponding points of the two hypersurfaces V^n, V^{*n} have the same local coordinates $x^1, \cdots, x^n$. Then §2 can be applied to the hypersurface V^n, and for the corresponding quantities for the hypersurface V^{*n} we shall use the same symbols with a star.

At first, we observe that the vector $Y_1 \times \cdots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \cdots \times Y_n$ is perpendicular to the normal vector N and that the n vectors $Y_1, \cdots, Y_n$ are linearly independent in the tangent hyperplane of the hypersurface V^n at the point P. Therefore the vector $Y_1 \times \cdots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \cdots \times Y_n$ can be written in the form

$$Y_1 \times \cdots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \cdots \times Y_n = a^{\alpha\beta} N_\beta. \tag{3.1}$$

Taking the scalar products of the both sides of equations (3.1) with the vector Y_γ and making use of equations (2.2), (2.3), (2.10), (2.23), we obtain

$$a^{\alpha\beta} b_{\beta\gamma} = g^{1/2} \delta^\alpha_\gamma \qquad (\alpha, \gamma = 1, \cdots, n). \tag{3.2}$$

Solving equations (3.2) for $a^{\alpha\beta}$ for each fixed α and substituting the results in equations (3.1), we are led to

$$Y_1 \times \cdots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \cdots \times Y_n = g^{1/2} b^{\alpha\beta} N_\beta. \tag{3.3}$$

Making use of equations (2.4), (2.9), (2.13), (2.22) and the relation

$$\begin{aligned} &Y_1 \times \cdots \times Y_{\beta-1} \times Y_{\beta\alpha} \times Y_{\beta+1} \times \cdots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \cdots \times Y_n \\ &\quad = - Y_1 \times \cdots \times Y_{\beta-1} \times N \times Y_{\beta+1} \times \cdots \times Y_{\alpha-1} \times Y_{\alpha\beta} \times Y_{\alpha+1} \times \cdots \times Y_n \\ &\qquad (\alpha > \beta;\ \alpha, \beta = 1, \cdots, n), \end{aligned}$$

it is easily seen that

$$\begin{aligned} &\sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha} (Y_1 \times \cdots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \cdots \times Y_n) \\ &= \sum_{\alpha=1}^{n} Y_1 \times \cdots \times Y_{\alpha-1} \times N_\alpha \times Y_{\alpha+1} \times \cdots \times Y_n = - n g^{1/2} M_1 N. \end{aligned} \tag{3.4}$$

Thus, from equations (3.3) and (3.4),

$$n g^{1/2} M_1 N = - \frac{\partial}{\partial x^\alpha} (g^{1/2} b^{\alpha\beta} N_\beta). \tag{3.5}$$

Taking the scalar products of the both sides of equation (3.5) with the position vector Y^* of the corresponding point P^* of the point P, we obtain in consequence of the relation $b^2 = fg$, obtained from equations (2.13) and (2.19),

$$n M_1 g^{1/2} p^* = \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} B^{\alpha\beta} N_\beta \cdot Y_\alpha^* - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha} \left(\frac{1}{f^{1/2}} B^{\alpha\beta} Y^* \cdot N_\beta \right), \tag{3.6}$$

where we have placed

$$p^* = Y^* \cdot N. \tag{3.7}$$

Integrating equation (3.6) with respect to $x^1, \cdots, x^n$ over the hypersurface V^n and applying the general Green's theorem (cf., for instance, [3, pp. 75–76]) to the second term on the right side of equation (3.6), we then obtain

$$\begin{aligned} \int_{V^n} n M_1 p^* dA &= \int_{V^n} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} B^{\alpha\beta} N_\beta \cdot Y_\alpha^* dx^1 \cdots dx^n \\ &\quad - \int_{V^{n-1}} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} (-1)^{\alpha-1} B^{\alpha\beta} Y^* \\ &\qquad \cdot N_\beta dx^1 \cdots dx^{\alpha-1} dx^{\alpha+1} \cdots dx^n. \end{aligned} \tag{3.8}$$

In order to use the formula (3.8) to derive an analogous expression for the integral $\int_{V^n} M_\alpha p^* dA$ for a general α ($\alpha = 1, \cdots, n$), in the space E^{n+1} we first consider a hypersurface $\overline{V}^n$ parallel to the hypersurface V^n so that the two hypersurfaces V^n, $\overline{V}^n$ have the same normals. It is evident that the vector equation of the hypersurface $\bar{V}^n$ can be written in the form

$$\bar{Y} = Y - tN, \tag{3.9}$$

where t is a real parameter. From equations (3.9), $N\cdot N=1$ and $N\cdot\bar{Y}_\alpha = N\cdot\partial\bar{Y}/\partial x^\alpha = 0$, it follows immediately that $\partial t/\partial x^\alpha = 0$ and therefore that t is constant. Making use of equations (2.8), (2.10), (2.15) and their analogous ones for the hypersurface $\bar{V}^n$, we obtain the coefficients of the first and the second fundamental forms of the hypersurface $\bar{V}^n$:

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + 2b_{\alpha\beta}t + f_{\alpha\beta}t^2, \tag{3.10}$$

$$\bar{b}_{\alpha\beta} = b_{\alpha\beta} + f_{\alpha\beta}t, \tag{3.11}$$

from which it follows easily by an elementary calculation that

$$\bar{b} = b\Delta, \tag{3.12}$$

$$\bar{g} = g\Delta^2, \tag{3.13}$$

$$\left| \bar{r}\bar{b}_{\alpha\beta} - \bar{g}_{\alpha\beta} \right| = \left| (\bar{r} - t)b_{\alpha\beta} - g_{\alpha\beta} \right| \Delta, \tag{3.14}$$

where $\bar{g} = \left|\bar{g}_{\alpha\beta}\right|$, $\bar{b} = \left|\bar{b}_{\alpha\beta}\right|$ and

$$\Delta = \left| \delta_\alpha^\beta + b_{\alpha\rho}g^{\rho\beta}t \right|, \tag{3.15}$$

$$\bar{r}_\alpha = 1/\bar{\kappa}_\alpha \qquad (\alpha = 1, \cdots, n), \tag{3.16}$$

$\bar{\kappa}_\alpha$ being the principal curvatures of the hypersurface $\bar{V}^n$. In consequence of equations (3.12), (3.13), (3.14) and (2.12), (2.13), (2.21) together with their analogues for the hypersurface $\bar{V}^n$, we have

$$\bar{M}_n d\bar{A} = M_n dA, \tag{3.17}$$

$$\bar{r}_\alpha = r_\alpha + t, \tag{3.18}$$

where $d\bar{A}$ is the area element of the hypersurface $\bar{V}^n$.

Similarly, let $\bar{V}^{*n}$ be a hypersurface in the space E^{n+1} parallel to the hypersurface V^{*n} and having the vector equation

$$\bar{Y}^* = Y^* - tN, \tag{3.19}$$

where t is the same arbitrary constant as in equation (3.9). For this one-to-one correspondence between the points of the two hypersurfaces $\bar{V}^n$, $\bar{V}^{*n}$, equation (3.8) can be written as, by means of equations (1.1) and (3.16),

$$\begin{aligned} &\int_{\bar{V}^n} \bar{p}^*\left(\sum \bar{r}_1\bar{r}_2 \cdots \bar{r}_{n-1}\right)\bar{M}_n d\bar{A} \\ &= \int_{\bar{V}^n} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} \bar{B}^{\alpha\beta}N_\beta\cdot\bar{Y}_\alpha^* dx^1 \cdots dx^n \\ &\quad - \int_{\bar{V}^{n-1}} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} (-1)^{\alpha-1}\bar{B}^{\alpha\beta}N_\beta\cdot\bar{Y}^* dx^1 \cdots dx^{\alpha-1}dx^{\alpha+1} \cdots dx^n, \end{aligned} \tag{3.20}$$

where $\bar{p}^* = \bar{Y}^* \cdot N = p^* - t$, $\bar{B}^{\alpha\beta}$ is the cofactor of $\bar{b}_{\alpha\beta}$ in $\bar{b}$, $\bar{Y}^*_\alpha = \partial \bar{Y}^*/\partial x^\alpha$, and $\bar{V}^{n-1}$ is the boundary of the hypersurface $\bar{V}^n$. Substitution, in equation (3.20), of equations (3.17), (3.18), (2.15) and the analogue of equation (2.10) for the hypersurface V^{*n} yields immediately

$$\begin{aligned}
&\int_{V^n} (p^* - t) \sum_{\alpha=1}^{n} (n - \alpha + 1)\left(\sum r_1 \cdots r_{\alpha-1}\right) t^{n-\alpha} M_n dA \\
(3.21)\qquad &= -\int_{V^n} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} \bar{B}^{\alpha\beta}(b^*_{\beta\alpha} + t f_{\alpha\beta}) dx^1 \cdots dx^n \\
&\quad + \int_{V^{n-1}} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} (-1)^{\alpha-1} Y^* \cdot N_\beta \bar{B}^{\alpha\beta} dx^1 \cdots dx^{\alpha-1} dx^{\alpha+1} \cdots dx^n,
\end{aligned}$$

which is an identity in t. Hence, by equating the coefficients of $t^0, t, \cdots, t^{n-1}$ on the both sides of equation (3.21) and using equation (1.1), we can obtain n formulas, one of which is equation (3.8). These n formulas have been obtained by the author [4] for the case where the two hypersurfaces V^n, V^{*n} coincide.

4. **Proof of the theorem.** In order to prove the theorem stated in the introduction, we may assume, for simplicity, that the local coordinate $x^1, \cdots, x^n$ of the two hypersurfaces V^n, V^{*n} be so chosen that

$$f_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta.$$

Then from equation (3.11) it follows that

$$(4.1)\qquad \bar{B}^{\alpha\beta} = \frac{f}{f_{\alpha\alpha} f_{\beta\beta}} \left[-b_{\beta\alpha} t^{n-2} + \sum_{\gamma=1, \gamma\neq\alpha, \gamma\neq\beta}^{n} (b_{\beta\gamma} b_{\gamma\alpha} - b_{\gamma\gamma} b_{\beta\alpha}) t^{n-3}/f_{\gamma\gamma} \right] + \cdots,$$

for $\beta \neq \alpha$,

$$\begin{aligned}
&\bar{B}^{\alpha\alpha} = \frac{f}{f_{\alpha\alpha}} \left[t^{n-1} + \sum_{\gamma=1, \gamma\neq\alpha}^{n} \frac{b_{\gamma\gamma}}{f_{\gamma\gamma}} t^{n-2} \right. \\
(4.2)\qquad &\left. \qquad + \frac{1}{2} \sum_{\beta,\gamma=1; \beta,\gamma\neq\alpha; \beta\neq\gamma}^{n} \frac{b_{\beta\beta} b_{\gamma\gamma} - b_{\beta\gamma}^2}{f_{\beta\beta} f_{\gamma\gamma}} t^{n-3} \right] + \cdots,
\end{aligned}$$

where the unwritten terms are of degrees $< n-3$ in t. Moreover, an elementary calculation from equation (2.18) leads to

$$(4.3)\qquad \sum r_1 r_2 = \frac{1}{2} \sum_{\alpha,\beta=1; \alpha\neq\beta}^{n} \frac{b_{\alpha\alpha} b_{\beta\beta} - b_{\alpha\beta}^2}{f_{\alpha\alpha} f_{\beta\beta}}.$$

Thus, by equating the coefficients of t^{n-2} on the both sides of equation (3.21) and using equations (4.1), (4.2), (4.3), we obtain

$$
\begin{aligned}
(n-1)\int_{V^n} p^*\Big(\sum_{\alpha=1}^{n} r_\alpha\Big) M_n dA
&= -\int_{V^n} \sum_{\alpha,\beta=1;\alpha\neq\beta}^{n} \frac{f^{1/2}}{f_{\alpha\alpha}f_{\beta\beta}} (b_{\beta\beta}\overset{*}{b}_{\alpha\alpha} - b_{\alpha\beta}\overset{*}{b}_{\beta\alpha}) dx^1 \cdots dx^n \\
&\quad + \int_{V^{n-1}} \sum_{\alpha,\beta=1;\alpha\neq\beta}^{n} (-1)^{\alpha-1} \frac{f^{1/2}}{f_{\alpha\alpha}f_{\beta\beta}} Y^* \\
&\quad \cdot (N_\alpha b_{\beta\beta} - N_\beta b_{\beta\alpha}) dx^1 \cdots dx^{\alpha-1} dx^{\alpha+1} \cdots dx^n.
\end{aligned} \tag{4.4}
$$

Replacing the hypersurface V^n by the hypersurface V^{*n} in equation (4.4) gives

$$
\begin{aligned}
(n-1)\int_{V^{*n}} p^*\Big(\sum_{\alpha=1}^{n} \overset{*}{r}_\alpha\Big) \overset{*}{M}_n dA^*
&= -\int_{V^n} \sum_{\alpha,\beta=1;\alpha\neq\beta}^{n} \frac{f^{1/2}}{f_{\alpha\alpha}f_{\beta\beta}} (\overset{*}{b}_{\beta\beta}\overset{*}{b}_{\alpha\alpha} - \overset{*}{b}_{\alpha\beta}\overset{*}{b}_{\beta\alpha}) dx^1 \cdots dx^n \\
&\quad + \int_{V^{n-1}} \sum_{\alpha,\beta=1;\alpha\neq\beta}^{n} (-1)^{\alpha-1} \frac{f^{1/2}}{f_{\alpha\alpha}f_{\beta\beta}} Y^* \\
&\quad \cdot (N_\alpha \overset{*}{b}_{\beta\beta} - N_\beta \overset{*}{b}_{\beta\alpha}) dx^1 \cdots dx^{\alpha-1} dx^{\alpha+1} \cdots dx^n.
\end{aligned} \tag{4.5}
$$

Since under the given one-to-one correspondence between the points of the two hypersurfaces V^n, V^{*n}, the two boundaries V^{n-1}, V^{*n-1} are congruent, we may assume that the corresponding points of the boundaries V^{n-1}, V^{*n-1} have the same local coordinates $u^1, \cdots, u^{n-1}$. Then the second fundamental forms of the boundaries V^{n-1}, V^{*n-1} corresponding to the common unit normal vector N of the hypersurfaces V^n, V^{*n} at the corresponding points of the boundaries V^{n-1}, V^{*n-1} are equal (see, for instance, [2, p. 192]), and therefore from equation (2.28) and its analogue for the hypersurface V^{*n} it follows that $b_{\alpha\beta} = \overset{*}{b}_{\alpha\beta}$ at corresponding points of the two boundaries V^{n-1}, V^{*n-1}. Thus the second integrals on the right side of equations (4.4), (4.5) are equal. On the other hand, by the assumption of the theorem we have $\sum_{\alpha=1}^{n} r_\alpha = \sum_{\alpha=1}^{n} \overset{*}{r}_\alpha$, and from equations (2.13), (2.19) and the analogous ones for the hypersurface V^{*n} it is seen at once that $M_n g^{1/2} = \overset{*}{M}_n g^{*1/2}$. Hence subtracting equation (4.4) from equation (4.5) yields

$$
\int_{V^n} \sum_{\alpha,\beta=1;\alpha\neq\beta}^{n} \frac{f^{1/2}}{f_{\alpha\alpha}f_{\beta\beta}} [(b_{\beta\beta}\overset{*}{b}_{\alpha\alpha} - b_{\alpha\beta}\overset{*}{b}_{\beta\alpha}) - (\overset{*}{b}_{\alpha\alpha}\overset{*}{b}_{\beta\beta} - \overset{*}{b}_{\alpha\beta}\overset{*}{b}_{\beta\alpha})] dx^1 \cdots dx^n = 0. \tag{4.6}
$$

Adding together equation (4.6) and the analogous one by interchanging the two hypersurfaces V^n, V^{*n}, we obtain

$$
\int_{V^n} \sum_{\alpha,\beta=1;\alpha\neq\beta}^{n} \frac{f^{1/2}}{f_{\alpha\alpha}f_{\beta\beta}} [(b_{\alpha\alpha} - \overset{*}{b}_{\alpha\alpha})(b_{\beta\beta} - \overset{*}{b}_{\beta\beta}) - (b_{\alpha\beta} - \overset{*}{b}_{\alpha\beta})^2] dx^1 \cdots dx^n = 0. \tag{4.7}
$$

From the assumption $\sum_{\alpha=1}^n r_\alpha = \sum_{\alpha=1}^n r_\alpha^*$, equation (2.19) and the analogous one for the hypersurface V^{*n}, we have

$$\sum_{\alpha=1}^{n} \frac{1}{f_{\alpha\alpha}} (b_{\alpha\alpha} - b_{\alpha\alpha}^*) = 0, \tag{4.8}$$

and therefore

$$\begin{aligned} &\sum_{\alpha,\beta=1;\alpha\neq\beta}^{n} \frac{1}{f_{\alpha\alpha}f_{\beta\beta}} (b_{\alpha\alpha} - b_{\alpha\alpha}^*)(b_{\beta\beta} - b_{\beta\beta}^*) \\ &\qquad = \sum_{\alpha,\beta=1}^{n} \frac{1}{f_{\alpha\alpha}f_{\beta\beta}} (b_{\alpha\alpha} - b_{\alpha\alpha}^*)(b_{\beta\beta} - b_{\beta\beta}^*) - \sum_{\alpha=1}^{n} \frac{1}{f_{\alpha\alpha}^2} (b_{\alpha\alpha} - b_{\alpha\alpha}^*)^2 \\ &\qquad = - \sum_{\alpha=1}^{n} \frac{1}{f_{\alpha\alpha}^2} (b_{\alpha\alpha} - b_{\alpha\alpha}^*)^2. \end{aligned}$$

Thus equation (4.7) is reduced to

$$\begin{aligned} \int_{V^n} \Bigg[&\sum_{\alpha=1}^{n} \frac{1}{f_{\alpha\alpha}^2} (b_{\alpha\alpha} - b_{\alpha\alpha}^*)^2 \\ &+ \sum_{\alpha,\beta=1;\alpha\neq\beta}^{n} \frac{1}{f_{\alpha\alpha}f_{\beta\beta}} (b_{\alpha\beta} - b_{\alpha\beta}^*)^2 \Bigg] f^{1/2} dx^1 \cdots dx^n = 0. \end{aligned} \tag{4.9}$$

It is obvious that the integrand of equation (4.9) is non-negative, and therefore equation (4.9) holds when and only when

$$b_{\alpha\beta} = b_{\alpha\beta}^* \qquad (\alpha, \beta = 1, \cdots, n), \tag{4.10}$$

from which, equations (2.11), (2.17), and the analogous ones for the hypersurface V^{*n} we obtain that $g_{\alpha\beta} = g_{\alpha\beta}^*$ $(\alpha, \beta = 1, \cdots, n)$. Hence the proof of the theorem is complete.

References

1. S. S. Chern, *Topics in differential geometry* (Mimeographed notes), Princeton, Institute for Advanced Study, 1951.
2. L. P. Eisenhart, *Riemannian geometry*, Princeton, 1926.
3. W. V. D. Hodge, *The theory and applications of harmonic integrals*, 2d ed., Cambridge, 1952.
4. C. C. Hsiung, *Some integral formulas for closed hypersurfaces*, Math. Scand. vol. 2 (1954) pp. 286–294.
5. ———, *A theorem on surfaces with a closed boundary*, Math. Zeit. vol. 64 (1956) pp. 41–46.
6. T. Kubota, *Über die Eibereiche im n-dimensionalen Raume*, Science Reports of the Tôhoku Imperial University (1) vol. 14 (1925) pp. 399–402.

Lehigh University,
Bethlehem, Pa.